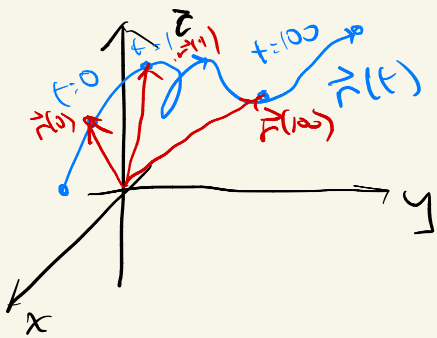
16.1 & 16.2

Line integrals and vector fields

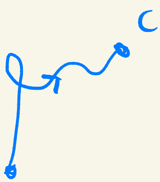
The big idea:

Parametrized curves allow us to talk about paths
e.g. path of a particle $\vec{r}(t)$ moving over time
- Each time t specifies a position vector $\vec{r}(t)$



Where are we going:

- Line integrals: Make sense of integrating a function $f(x,y,z)$ over paths.
  Might encode something meaningful



Ex: arc length $f(x,y,z)=1$

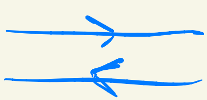
$$\int_C f(x,y,z)\,ds = \int_a^b f(\overbrace{\vec{r}(t)}^{\text{param.}}) \cdot \overbrace{|\vec{r}\,'(t)|\,dt}^{ds}$$

picked parametrization for C

$$= \int_a^b 1 \cdot |\vec{r}\,'(t)|\,dt$$

since $f(x,y,z)=1$ for arc length

orientation means specify coordinate system
(in 1D, specify where t is increasing) knowing orientation



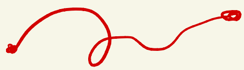
= → and specifying, ← otherwise negative

- Ex: work

Recall: work = $\vec{\text{Force}} \cdot \vec{\text{distance}}$

C

How to generalize to



C

$\underbrace{f(x,y,z)}_{\text{scalar function}}$ represents a force

that will depend upon "vector field" $\vec{F}$

choose parametrization

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|\,dt$$

- Ex: Flux

Line Integration:

Basically the same as arclength

curve C (oriented)

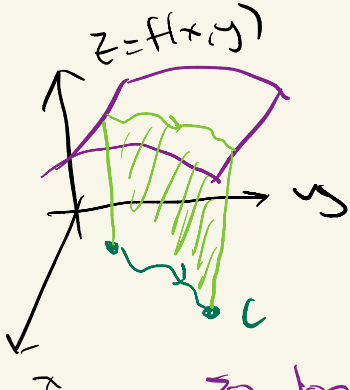
$$ds = |\vec{r}\,'(t)|\,dt$$

Riemann Sum

$$\text{Area} = \int_C f(x,y)\,ds$$

chosen param $\vec{r}(t)$

$$= \int_a^b f(\vec{r}(t)) \cdot \underbrace{\underbrace{|\vec{r}\,'(t)| \cdot dt}_{ds}}_{\text{scalar}\cdot dt}$$



So long as our parametrizations

$$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$$

If g, h, and k are smooth functions (at least differentiable), this works

Ex:

Integrate $f(x,y,z) = x - 3y^2 + z$ over the line segment C joining the origin to the point (1,1,1)

Sol'n:

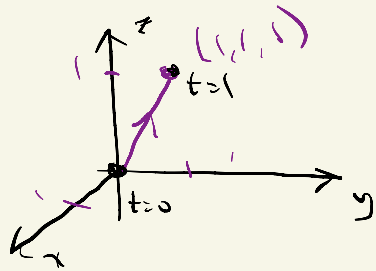
From E.C. Tuesday: can find parametrization

$0 \le t \le 1$

$$\vec{r}(t) = \underset{\text{starting point } t=0}{\vec{u}} + t(\underbrace{\vec{v}}_{\text{ending point } t=1} - \vec{u})$$

$$= (0,0,0) + t((1,1,1) - (0,0,0))$$

$$= t(1,1,1)$$



Now we can integrate:

$$\int_C f(x,y,z)\,ds = \int_0^1 f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|\,dt$$

$\vec{r}(t) = t(1,1,1)$

$$= \int_0^1 f(t,t,t) \cdot |(1,1,1)|\,dt$$

$$= \int_0^1 (t - 3t^2 + t) \cdot \sqrt{3}\,dt$$

$$= \sqrt{3} \int_0^1 2t - 3t^2\,dt$$

$$= 0$$

Same properties as regular integral:

① Linearity. If $\alpha \in \mathbb{R}$, f, g are functions

$$\int_C \alpha f + g\,ds = \alpha \int_C f\,ds + \int_C g\,ds$$

set of real numbers

$\alpha \in \mathbb{R}$ = α is a real number

belongs to

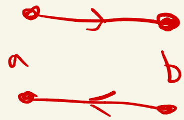
② Domination. If $f(x,y,z) \geq g(x,y,z)$ for all $(x,y,z) \in C$

$$\int_C f\,ds \geq \int_C g\,ds$$

③ Additivity (this is why curves need orientation!)

Recall MAT21B:

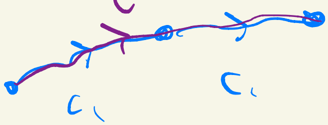
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$



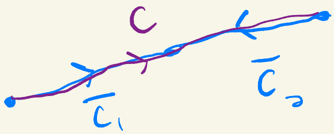
(in book: $C = C_1 \cup C_2$ but this misses orientation!)

If $C = C_1 + C_2$, then

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$$



$C = C_1 + C_2$



$\bar{C} = \bar{C}_1 - \bar{C}_2$

- To add, they must be touching!
- In terms of parametrization

$C \Leftrightarrow \vec{r}(t)$

$-C$

$\Leftrightarrow \vec{r}(-t)$

- Note:

C

$C + (-C) = 0$

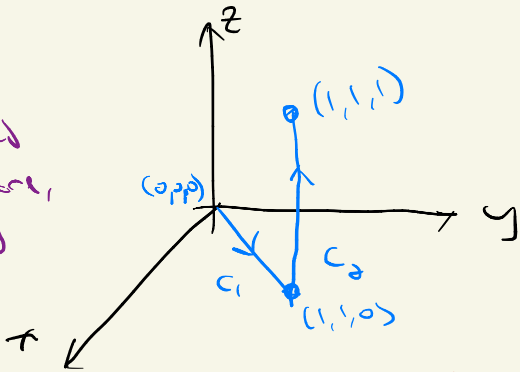
$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0$$

Ex:

Same function $f(x,y,z) = x - 3y^2 + z$ only

$C_1 + C_2$:

(notice: same start and end points as before, but different path)



$\vec{r}_1(t) = \vec{u} + t(\vec{v} - \vec{u})$
$= (0,0,0) + t((1,1,0) - (0,0,0))$
$= t(1,1,0)$

Soln:

Now need 2 params:

C_1: $\vec{r}_1(t) = t\hat{\imath} + t\hat{\jmath} = (t,t,0)$

$0 \leq t \leq 1$

C_2: $\vec{r}_2(t) = \hat{\imath} + \hat{\jmath} + t\hat{k} = (1,1,t)$

$0 \leq t \leq 1$

$\boxed{10:55}$

additivity

$$\Rightarrow \int_{C_1 + C_2} f\,ds \stackrel{*}{=} \int_{C_1} f\,ds + \int_{C_2} f\,ds$$

$$= \int_0^1 f(t,t,0)\cdot|(1,1,0)|\,dt + \int_0^1 f(1,1,t)\cdot|(0,0,1)|\,dt$$

$$= \int_0^1 (t - 3t^2 + 0)\cdot\sqrt{2}\,dt + \int_0^1 (1 - 3 + t)\cdot 1\,dt$$

$$= -\frac{\sqrt{2}}{2} - \frac{3}{2}$$

This is not 0, the earlier integral! Line integrals can be path-dependent!

16.2: Vector Fields

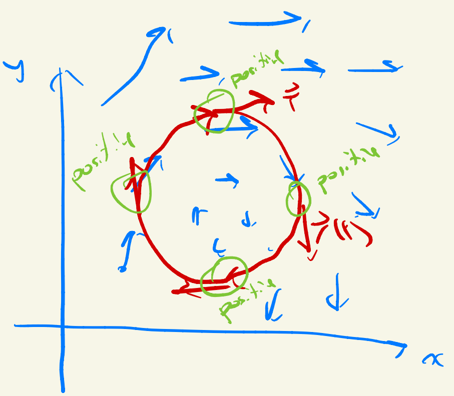
Now... where do interesting scalar functions to line integrate come from?

Ex:

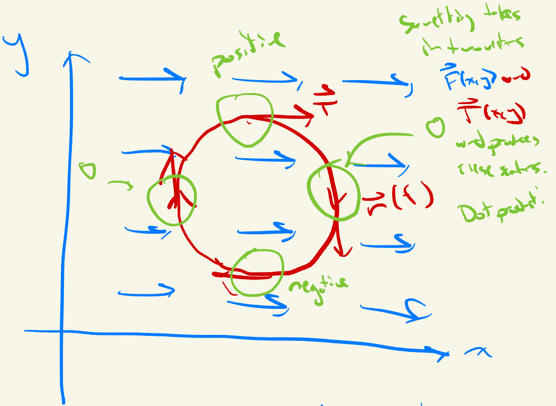
Imagine a swimmer $\vec{r}(t)$ in the ocean and a lazy river.

We want to model the (work) energy done by the ocean on him.

In water, each point (x,y) has water flowing in a direction: "attach" a vector to each point to represent this current



ocean



lazy river

These are vector fields $\vec{F}(x,y)$.

For each (x,y) coordinate, we attach a vector

$$\vec{F}(x,y) = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$$

N, M are scalar functions

Vector Field

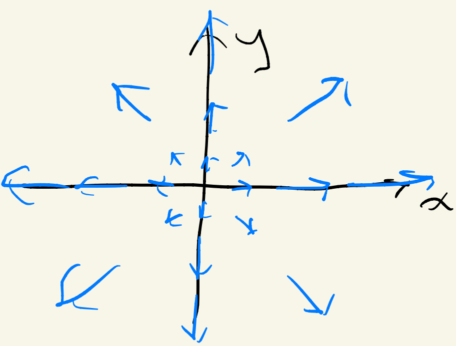
$$\vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$$

"Every coordinate vector (x,y,z) gets a vector (M,N,P)"

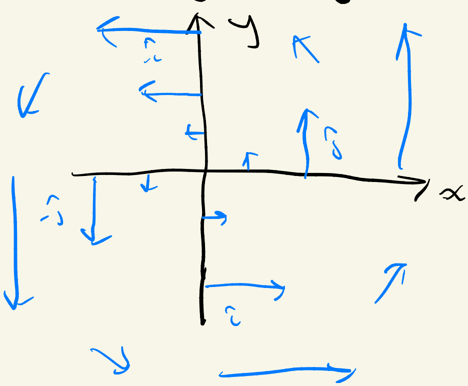
- $\vec{F}$ is <u>continuous</u> means M, N, P (each of which is a scalar function) are all continuous
- F is <u>differentiable</u> means M, N, P all differentiable.

Ex:

$$\vec{F} = x\,\hat{i} + y\,\hat{j}$$

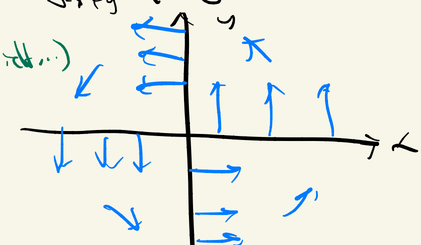


$$\vec{F} = -y\,\hat{i} + x\,\hat{j}$$



F is a unit vector at every point →

$$\vec{F} = \frac{1}{\sqrt{x^2+y^2}}\left(-y\,\hat{i} + x\,\hat{j}\right)$$



Commonly...
- Force fields (gravity field, electric field...)
- velocity field (water velocity, population movement)
- Gradients

Def: Gradient

If a scalar function $f(x,y,z)$ is differentiable, then the gradient field of f is

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

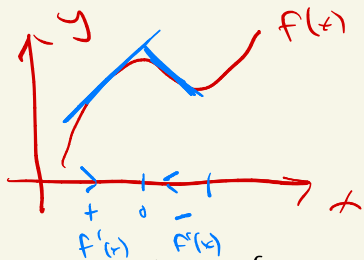
"del" or "nabla"

contains all derivative info

"multidimensional analogue of $\frac{d}{dx}$"

- Direction of $\nabla f(x,y,z)$ gives direction of "steepest ascent" of $f(x,y,z)$

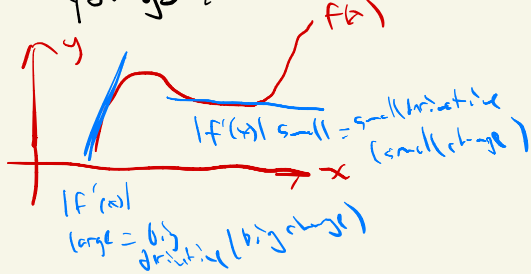
Recall 21A: derivative



positive = increase x to grow f(x)

negative = decrease x to grow f(x)

- Magnitude of $\nabla f(x,y,z)$ gives "directional derivative" in that direction
  - If you move in that direction, how fast will you go?

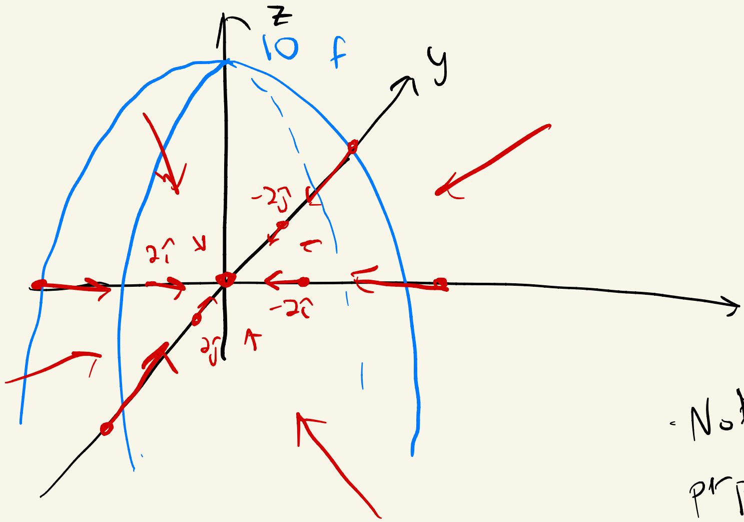


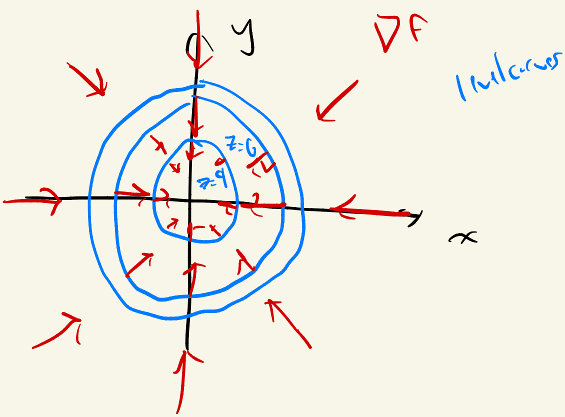
Ex:

$$f(x,y) = 10 - x^2 - y^2 \quad \bullet\ f$$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$= -2x\,\hat{i} - 2y\,\hat{j} \quad \bullet\ \nabla f$$





- Notice: ∇f is always perpendicular to level set/curves. (if we move along level set, no change in f

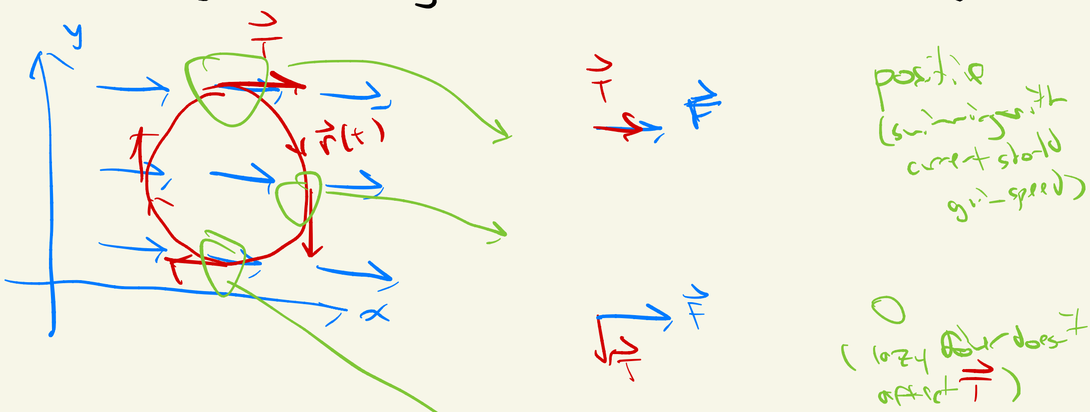
  $\Rightarrow$ All partial derivatives are 0

  $\Rightarrow$ $\nabla f = \vec{0}$

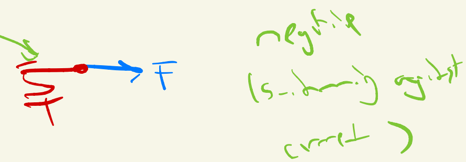
- steeper incline $=$ bigger $|\nabla f|$

Now we have vector fields.

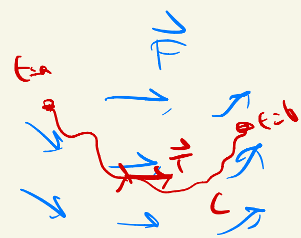
How do we get interesting/useful scalar functions to integrate



Need something that turns vectors at every point at coordinate to scalar measuring this picture



⟹ Dot product at every point!



Def: Line integral of $\vec F$

Let $\vec F$ be a continuous vector field and C a smooth curve parametrized by $\vec r(t)$

The line integral of $\vec F$ along C is

$$\int_C \vec F \cdot d\vec s := \int_C \vec F \cdot \frac{d\vec r}{ds}\, ds = \int_C \vec F \cdot \vec T\, ds$$

$\vec T$ unit tangent to C

Ex:

$$F(x,y,z) = z\hat{i} + xy\hat{j} - y^2\hat{k}$$

$$C: \vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}$$

$$0 \le t \le 1$$

Find $\int_C \vec{F} \cdot d\vec{r}$

Soln:

$$\vec{F}(\vec{r}(t)) = \vec{F}(t^2, t, \sqrt{t})$$

$$= \sqrt{t}\,\hat{i} + t^3\hat{j} - t^2\hat{k}$$

$$\frac{d\vec{r}}{dt} = 2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\,dt$$

$$= \int_0^1 \left(2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}\right)$$

$$= \frac{17}{20}$$